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ON $(p, q)^{th}$ GOL'DBERG ORDER AND $(p, q)^{th}$ GOL'DBERG TYPE OF AN ENTIRE FUNCTION OF SEVERAL COMPLEX VARIABLES REPRESENTED BY MULTIPLE DIRICHLET SERIES

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Abstract: Introducing the idea of $(p, q)^{th}$ Gol'dberg order and $(p, q)^{th}$ Gol'dberg type of an entire function f of several complex variables in a domain D we generalise some earlier results.

Keywords and Phrases: Entire function, Multiple Dirichlet series, Gol'dberg order, Gol'dberg type.

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1. Introduction

In this paper we denote complex and real n-space by $\mathbb{C}^n$ and $\mathbb{R}^n$ respectively. We write the elements $(s_1, s_2, ..., s_n)$, $(Re\ s_1, Re\ s_2, ..., Re\ s_n)$, $(\sigma_1, \sigma_2, ..., \sigma_n)$, $(m_1, m_2, ..., m_n)$ etc. of $\mathbb{C}^n$ by their corresponding unsuffixed symbols s, $Re\ s$, σ, m etc. respectively.
For $x, y \in \mathbb{C}^n$, we define $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$, $xy = (x_1y_1, x_2y_2, ..., x_ny_n)$, $||x|| = x_1 + x_2 + ... + x_n$, $x + r = (x_1 + r, x_2 + r, ..., x_n + r)$ for $r \in \mathbb{R}$. By I^n we shall mean the Cartesian product of n copies of I where I is the set of non-negative integers. For $k \in I$, $\bar{k}$ will denote the real n-tuple $(k, k, ..., k)$. For an entire function f with domain $\mathbb{C}^n$, f^k will denote the function $\frac{\partial^{||k||} f}{\partial_{s_1}^{k_1} ... \partial_{s_n}^{k_n}}$, where $k \in I^n$ and $f^{(\bar{0})} = f$.
Consider the multiple Dirichlet series

$$f(s_1, s_2, ..., s_n) = \sum_{m_1, m_2, ..., m_n = 1}^{\infty} a_{m_1, m_2, ..., m_n} exp(s_1\lambda_{1m_1} + s_2\lambda_{2m_2} + ... + s_n\lambda_{nm_n})$$

$$i.e., f(s) = \sum_{m=1}^{\infty} a_m exp||s\lambda_{nm_n}||, \quad (s_j = \sigma_j + it_j, \quad j = 1, 2, .., n) \tag{1.1}$$

where $a_m \in \mathbb{C}$, λ_{nm_n} denotes the real-tuple $(\lambda_{1m_1}, \lambda_{2m_2}, ..., \lambda_{nm_n})$; $0 \leq \lambda_{p_1} < \lambda_{p_2} < ... < \lambda_{p_k} \to \infty \ as \ k \to \infty, \ for \ p = 1, 2, ..., n.$
Janusaukas [2] had shown that if there exists a tuple $p > \bar{0} = (0, 0, ..., 0)$ such that

$$\limsup_{||m|| \to \infty} \frac{\sum_{k=1}^{n} logm_k}{||p\lambda_{nm_n}||} = 0 \tag{1.2}$$

then the domain of absolute convergence of the series (1.1) coincides with its domain of convergence.
The necessary and sufficient [4] condition that the series (1.1) satisfying (1.2) to be entire is that

$$\limsup_{||m|| \to \infty} \frac{log|a_m|}{||\lambda_{nm_n}||} = -\infty. \tag{1.3}$$

For two entire functions f and g, Hadamard product [6] $f * g$ is defined by

$$f(s) * g(s) = \sum_{m=1}^{\infty} a_m b_m exp||s\lambda_{nm_n}|| \tag{1.4}$$

where $f(s) = \sum_{m=1}^{\infty} a_m exp||s\lambda_{nm_n}||$ and $g(s) = \sum_{m=1}^{\infty} b_m exp||s\lambda_{nm_n}||$.
For $k \in I^n$, we define

$$f^k(s) = \sum_{m=1}^{\infty} \lambda^k{}_{nm_n} a_m exp||s\lambda_{nm_n}|| \tag{1.5}$$

and

$$f^k(s) * g^k(s) = \sum_{m=1}^{\infty} \lambda^{2k}{}_{nm_n} a_m b_m exp||s\lambda_{nm_n}||. \tag{1.6}$$

Following Sato [3], we write $log^{[0]}x = x, exp^{[0]}x = x$ and for positive integer $m \geq 1$, $log^{[m]}x = log(log^{[m-1]}x)$, $exp^{[m]}x = exp(exp^{[m-1]}x)$.
In this paper F stands for the family of all multiple Dirichlet series of the form (1.1) satisfying (1.2) and (1.3). Then $f \in F$ denotes an entire function over $\mathbb{C}^n$. For given $l \in \mathbb{R}^n$, Sarkar [4] defined the poly half plane D as $D = \{s : s \in \mathbb{C}^n, \ Re \ s = \sigma \leq l\}$. Then the region $D + r$ depending on the parameter $r \in R$ is defined as

$D+r=\{s+r : s\in D\}$. For any $f\in F$, Sarkar [4] defined the maximum modulus $M_{f,D}(r)$ with respect to the region D where $r\in R$ as

$$M_{f,D}(r) = sup\{|f(s)| : s\in D+r\}.$$

Also the maximum term $\mu_f=\mu_f(\sigma)$ at $\sigma\in\mathbb{R}^n$ is defined by

$$\mu_f(\sigma) = \sup_{m\in\mathbb{N}^n}\{|a_m|exp||\sigma\lambda_{nm_n}||\}$$

where $\mathbb{N}$ is the set of all positive integers.

Definition 1.1. *[4] The Gol'dberg order $\rho(D)$ of f with respect to the domain D is defined as*

$$\rho(D) = \limsup_{r\to\infty}\frac{loglogM_{f,D}(r)}{r}.$$

Definition 1.2. *[4] The Gol'dberg order $\rho_k(D)$ of f^k with respect to the domain D is defined as*

$$\rho_k(D) = \limsup_{r\to\infty}\frac{loglogM_{f^k,D}(r)}{r}.$$

Definition 1.3. *[4] The Gol'dberg type $\tau(D)$ of f with respect to the domain D is defined as*

$$\tau(D) = \limsup_{r\to\infty}\frac{logM_{f,D}(r)}{e^{r\rho(D)}},\ \ if\ \rho(D)>0.$$

Definition 1.4. *[4] $f\in F$ is of Gol'dberg order ρ iff $\rho=\rho(D)=\limsup_{||m||\to\infty}$ $\dfrac{||\lambda_{nm_n}||\ log||\lambda_{nm_n}||}{-log\{|a_m|\phi_D(m)\}}$ where $\phi_D(m)=\sup_{s\in D}|\ exp||s\lambda_{nm_n}||\ |$.*

In [4], Sarkar proved the following theorems.

Theorem 1.1. *Let $f\in F$. Then*

$$\mu_f(l+\sigma)\le M_{f,D}(\sigma)\le K\mu_f(l+\sigma+\epsilon)$$

where K is positive constant depending on $\epsilon\in R_+$.

Theorem 1.2. *Let $f\in F$. Then for any $K\in R$*
(i) $\rho(D+K)=\rho(D)$
(ii) If $\rho(D)>0$, then $\tau(D+K)=e^{K\rho}\ \tau(D)$ where $\rho=\rho(D)$.

In [1], Alam proved the following theorems.

Theorem 1.3. *The function* $f^k * g^k$ *as defined (1.6) is an entire function.*

Theorem 1.4. *Let* f *and* g *be entire functions where*

$$f^k(s) = \sum_{m=1}^{\infty} \lambda_{nm_n}^k a_m exp\{s\lambda_{nm_n}\}$$

and

$$g^k(s) = \sum_{m=1}^{\infty} \lambda_{nm_n}^k b_m exp\{s\lambda_{nm_n}\}$$

having G-order ρ_{k_f} $(0 < \rho_{k_f} < \infty)$ *and* ρ_{k_g} $(0 < \rho_{k_g} < \infty)$ *respectively. Then* $f^k(s)*g^k(s)$ *is an entire function with G-order* ρ_k *such that* $\rho_k \le (\rho_{k_f}\rho_{k_g})^{\frac{1}{2}}$ *provided that*

$$log\frac{1}{|\lambda_{nm_n}^{2k} a_m b_m|} \sim [log\frac{1}{|\lambda_{nm_n}^k a_m|} log\frac{1}{|\lambda_{nm_n}^k b_m|}]^{\frac{1}{2}}.$$

Theorem 1.5. *Let* f^k *and* g^k *be entire functions of G-order* ρ_{k_f} $(0 < \rho_{k_f} < \infty)$ *and* ρ_{k_g} $(0 < \rho_{k_g} < \infty)$ *and finite G-type* $T_{k_f}(D)$ *and* $T_{k_g}(D)$ *respectively having the same fundamental domain D. If* $f^k * g^k$ *is of G-order* ρ_k $(0 < \rho_k < \infty)$*, where* $logM_{f^k*g^k}(r) \sim logM_{f^k,D}(r) logM_{g^k,D}(r)$*, then* $\rho_k \le \rho_{k_f} + \rho_{k_g}$.
Also if $T_k(D)$ *be the G-type of* $f^k * g^k$ *with respect to the domain D then* $T_k(D) \le T_{k_f}(D)T_{k_g}(D)$ *provided the sign of equality holds in* $\rho_k \le \rho_{k_f} + \rho_{k_g}$.
In [5], Singh and Rastogi proved the following theorems.

Theorem 1.6. *Let* f *be an entire function defined in domain D and* $k \in R$*. Then*
(i) $\rho^q(D + k) = \rho^q(D)$
(ii) If $\rho^q(D) > 0$*, then* $T^q(D + k) = T^q(D)$.

Theorem 1.7. *Let* f *and* g *be entire functions, where*

$$f^k(s) = \sum_{m=1}^{\infty} \lambda_{nm_n}^k a_m exp\{s\lambda_{nm_n}\}$$

and

$$g^k(s) = \sum_{m=1}^{\infty} \lambda_{nm_n}^k b_m exp\{s\lambda_{nm_n}\}$$

having q^{th} *Gol'dberg order* $\rho_{k_f}^q$ $(0 < \rho_{k_f}^q < \infty)$ *and* $\rho_{k_g}^q$ $(0 < \rho_{k_g}^q < \infty)$ *respectively. Then* $f^k(s) * g^k(s)$ *is an entire function with* q^{th} *Gol'dberg order* ρ_k^q *such that* $\rho_k^q \le$

$(\rho^q_{k_f}\rho^q_{k_g})^{\frac{1}{2}}$ *provided*

$$log\frac{1}{|\lambda^{2k}_{nm_n}a_m b_m|} \sim [log\frac{1}{|\lambda^{k}_{nm_n}a_m|}log\frac{1}{|\lambda^{k}_{nm_n}b_m|}]^{\frac{1}{2}}.$$

Theorem 1.8. *Let f^k and g^k be entire functions with q^{th} Gol'dberg order $\rho^q_{k_f}$ $(0<\rho^q_{k_f}<\infty)$ and $\rho^q_{k_g}$ $(0<\rho^q_{k_g}<\infty)$ respectively and also q^{th} Gol'dberg type $T^q_{k_f}$ and $T^q_{k_g}$. Then $\rho^q_k \le \rho^q_{k_f}+\rho^q_{k_g}$ and also $T^q_k \le T^q_{k_f}(D)T^q_{k_g}(D)$.*

Theorem 1.9. *If*

$$\rho^q(D) = \limsup_{\sigma\to\infty}\frac{log^{[q]}M_{f,D}(\sigma)}{log\sigma}$$

then

$$\limsup_{\sigma\to\infty}\frac{log^{[q]}\mu_{f,D}(l+\sigma)}{log\sigma} \le \rho$$

and

$$\limsup_{\sigma\to\infty}\frac{log^{[q]}\mu_{f,D}(l+\sigma+\epsilon)}{log\sigma} \ge \frac{\rho}{k},$$

where k is a positive constant.

With this in view we first introduce the following definitions.

Definition 1.5. *The $(p,q)^{th}$ Gol'dberg order of an entire function f in the corresponding domain D is defined by*

$$\rho^{[p,q]}(D) = \limsup_{\sigma\to\infty}\frac{log^{[p]}M_{f,D}(\sigma)}{log^{[q]}\sigma}.$$

Definition 1.6. *The $(p,q)^{th}$ Gol'dberg type of an entire function f in the corresponding domain D is defined by*

$$T^{[p,q]}(D) = \limsup_{\sigma\to\infty}\frac{log^{[p-1]}M_{f,D}(\sigma)}{[log^{[q-1]}\sigma]^{\rho^{[p,q]}(D)}}.$$

Definition 1.7. *The $(p,q)^{th}$ Gol'dberg order of an entire function f^k in the corresponding domain D is defined by*

$$\rho^{[p,q]}_{k_f}(D) = \limsup_{\sigma\to\infty}\frac{log^{[p]}M_{f^k,D}(\sigma)}{log^{[q]}\sigma}. \tag{1.7}$$

Definition 1.8. *The $(p,q)^{th}$ Gol'dberg type of an entire function f^k in the corresponding domain D is defined by*

$$T_{k_f}^{[p,q]}(D) = \limsup_{\sigma\to\infty} \frac{log^{[p-1]} M_{f^k,D}(\sigma)}{[log^{[q-1]}\sigma]^{\rho_{k_f}^{[p,q]}(D)}}. \tag{1.8}$$

2. Main Results

With these definitions of $(p,q)^{th}$ Gol'dberg order and $(p,q)^{th}$ Gol'dberg type we prove above theorems in this direction.

Theorem 2.1. *Let f be an entire function defined in domain D and $K \in R$. Then*
(i) $\rho^{[p,q]}(D+K) = \rho^{[p,q]}(D)$
(ii) If $\rho^{[p,q]}(D) > 0,$ *then* $T^{[p,q]}(D+K) = T^{[p,q]}(D)$.

Proof. (i) Let $K \in R$. Then from the definition of $\rho^{[p,q]}(D)$ we write

$$\begin{aligned}\rho^{[p,q]}(D) &= \limsup_{\sigma\to\infty} \frac{log^{[p]} M_{f,D}(\sigma+K)}{log^{[q]}(\sigma+K)} \\ &= \limsup_{\sigma\to\infty} \frac{log^{[p]} M_{f,D+K}(\sigma)}{log^{[q]}(\sigma)} \frac{log^{[q]}(\sigma)}{log^{[q]}(\sigma+K)} \\ &= \rho^{[p,q]}(D+K).\end{aligned}$$

(ii) By the definition of $T^{[p,q]}(D)$ we write

$$\begin{aligned}T^{[p,q]}(D) &= \limsup_{\sigma\to\infty} \frac{log^{[p-1]} M_{f,D}(\sigma+K)}{[log^{[q-1]}(\sigma+K)]^{\rho^{[p,q]}(D)}} \\ &= \limsup_{\sigma\to\infty} \frac{log^{[p-1]} M_{f,D+K}(\sigma)}{[log^{[q-1]}\sigma]^{\rho^{[p,q]}(D)}} \frac{[log^{[q-1]}\sigma]^{\rho^{[p,q]}(D)}}{[log^{[q-1]}(\sigma+K)]^{\rho^{[p,q]}(D)}} \\ &= T^{[p,q]}(D+K).\end{aligned}$$

Remark 2.1. *From Theorem 2.1, it is clear that the $(p,q)^{th}$ Gol'dberg order and $(p,q)^{th}$ Gol'dberg type of $f \in F$ are independent of the choice of the poly half plane D.*

Theorem 2.2. *Let f and g be entire functions, where*

$$f^k(s) = \sum_{m=1}^{\infty} \lambda_{nm_n}^k a_m exp\{s\lambda_{nm_n}\}$$

and

$$g^k(s) = \sum_{m=1}^{\infty} \lambda_{nm_n}^k b_m exp\{s\lambda_{nm_n}\}$$

having $(p,q)^{th}$ Gol'dberg order $\rho_{k_f}^{[p,q]}$ $(0<\rho_{k_f}^{[p,q]}<\infty)$ and $\rho_{k_g}^{[p,q]}$ $(0<\rho_{k_g}^{[p,q]}<\infty)$ respectively. Then $f^k(s)*g^k(s)$ is an entire function with $(p,q)^{th}$ Gol'dberg order $\rho_k^{[p,q]}$ such that $\rho_k^{[p,q]}\le(\rho_{k_f}^{[p,q]}.\rho_{k_g}^{[p,q]})^{\frac{1}{2}}$ provided

$$log^{[p]}M_{f^k*g^k,D}(\sigma)\sim[log^{[p]}M_{f^k,D}(\sigma).log^{[p]}M_{g^k,D}(\sigma)]^{\frac{1}{2}}.$$

Proof. We have $f^k(s)*g^k(s)$ is an entire function by Theorem 1.3.
Now from the definition of $(p,q)^{th}$ Gol'dberg order of entire functions f^k and g^k we write,

$$\rho_{k_f}^{[p,q]}=\limsup_{\sigma\to\infty}\frac{log^{[p]}M_{f^k,D}(\sigma)}{log^{[q]}\sigma}$$

and

$$\rho_{k_g}^{[p,q]}=\limsup_{\sigma\to\infty}\frac{log^{[p]}M_{g^k,D}(\sigma)}{log^{[q]}\sigma}.$$

Hence for an arbitrary $\epsilon>0$,

$$\frac{log^{[p]}M_{f^k,D}(\sigma)}{log^{[q]}\sigma}\le(\rho_{k_f}^{[p,q]}+\frac{\epsilon}{2})$$

and

$$\frac{log^{[p]}M_{g^k,D}(\sigma)}{log^{[q]}\sigma}\le(\rho_{k_g}^{[p,q]}+\frac{\epsilon}{2}).$$

Now taking product we get,

$$\frac{log^{[p]}M_{f^k,D}(\sigma).log^{[p]}M_{g^k,D}(\sigma)}{(log^{[q]}\sigma)^2}\le(\rho_{k_f}^{[p,q]}+\frac{\epsilon}{2})(\rho_{k_g}^{[p,q]}+\frac{\epsilon}{2})$$

or,

$$\frac{[log^{[p]}M_{f^k,D}(\sigma).log^{[p]}M_{g^k,D}(\sigma)]^{\frac{1}{2}}}{(log^{[q]}\sigma)}\le[(\rho_{k_f}^{[p,q]}+\frac{\epsilon}{2})(\rho_{k_g}^{[p,q]}+\frac{\epsilon}{2})]^{\frac{1}{2}}.$$

Now if

$$log^{[p]}M_{f^k*g^k,D}(\sigma)\sim[log^{[p]}M_{f^k,D}(\sigma).log^{[p]}M_{g^k,D}(\sigma)]^{\frac{1}{2}}$$

then we get from above

$$\frac{log^{[p]}M_{f^k*g^k,D}(\sigma)}{log^{[q]}\sigma}\le[(\rho_{k_f}^{[p,q]}+\frac{\epsilon}{2})(\rho_{k_g}^{[p,q]}+\frac{\epsilon}{2})]^{\frac{1}{2}}.$$

Therefore,

$$\limsup_{\sigma\to\infty}\frac{log^{[p]}M_{f^k*g^k,D}(\sigma)}{log^{[q]}\sigma}\le[\rho_{k_f}^{[p,q]}.\rho_{k_g}^{[p,q]}]^{\frac{1}{2}}.$$

Hence $\rho_k^{[p,q]} \le (\rho_{k_f}^{[p,q]}.\rho_{k_g}^{[p,q]})^{\frac{1}{2}}$.

Theorem 2.3. *Let $f^k(s)$ and $g^k(s)$ be entire functions with $(p,q)^{th}$ Gol'dberg order $\rho_{k_f}^{[p,q]}$ $(0<\rho_{k_f}^{[p,q]}<\infty)$ and $\rho_{k_g}^{[p,q]}$ $(0<\rho_{k_g}^{[p,q]}<\infty)$ and finite $(p,q)^{th}$ Gol'dberg type $T_{k_f}^{[p,q]}$ and $T_{k_g}^{[p,q]}$ respectively having the same fundamental domain D. If $f^k(s) * g^k(s)$ is of $(p,q)^{th}$ Gol'dberg order $\rho_k^{[p,q]}$ $(0<\rho_k^{[p,q]}<\infty)$, where $[log^{[p-1]}M_{f^k,D}(\sigma)\ log^{[p-1]}M_{g^k,D}\ (\sigma)] \sim log^{[p-1]}\ M_{f^k*g^k,D}(\sigma)$, then $\rho_k^{[p,q]} \le \rho_{k_f}^{[p,q]} + \rho_{k_g}^{[p,q]}$.*
Also if $T_k^{[p,q]}$ be the $(p,q)^{th}$ Gol'dberg type of $f^k(s)*g^k(s)$ with respect to the domain D then $T_k^{[p,q]} \le T_{k_f}^{[p,q]}T_{k_g}^{[p,q]}$ provided the sign of equality holds in $\rho_k^{[p,q]} \le \rho_{k_f}^{[p,q]} + \rho_{k_g}^{[p,q]}$.

Proof. We have

$$\rho_{k_f}^{[p,q]} = \limsup_{\sigma\to\infty} \frac{log^{[p]}M_{f^k,D}(\sigma)}{log^{[q]}\sigma}$$

and

$$\rho_{k_g}^{[p,q]} = \limsup_{\sigma\to\infty} \frac{log^{[p]}M_{g^k,D}(\sigma)}{log^{[q]}\sigma}.$$

Hence for an arbitrary $\epsilon > 0$,

$$\frac{log^{[p]}M_{f^k,D}(\sigma)}{log^{[q]}\sigma} \le (\rho_{k_f}^{[p,q]} + \frac{\epsilon}{2})$$

$$or,\ log^{[p]}M_{f^k,D}(\sigma) \le (\rho_{k_f}^{[p,q]} + \frac{\epsilon}{2})\ log^{[q]}\sigma$$

$$or,\ log^{[p-1]}M_{f^k,D}(\sigma) \le exp[(\rho_{k_f}^{[p,q]} + \frac{\epsilon}{2})\ log^{[q]}\sigma]$$

and similarly

$$log^{[p-1]}M_{g^k,D}(\sigma) \le exp[(\rho_{k_g}^{[p,q]} + \frac{\epsilon}{2})\ log^{[q]}\sigma].$$

Now taking product we get,

$$log^{[p-1]}M_{f^k,D}(\sigma)\ log^{[p-1]}M_{g^k,D}(\sigma) \le exp[(\rho_{k_f}^{[p,q]} + \rho_{k_g}^{[p,q]} + \epsilon)\ log^{[q]}\sigma].$$

Thus if

$$[log^{[p-1]}M_{f^k,D}(\sigma)\ log^{[p-1]}M_{g^k,D}(\sigma)] \sim log^{[p-1]}M_{f^k*g^k,D}(\sigma)$$

then we get from above

$$log^{[p-1]}M_{f^k*g^k,D}(\sigma) < exp[(\rho_{k_f}^{[p,q]} + \rho_{k_g}^{[p,q]} + \epsilon)\ log^{[q]}\sigma].$$

Therefore,

$$\limsup_{\sigma\to\infty}\frac{log^{[p]}M_{f^k*g^k,D}(\sigma)}{log^{[q]}\sigma}\leq[\rho_{k_f}^{[p,q]}+\rho_{k_g}^{[p,q]}]+\epsilon.$$

Hence $\rho_k^{[p,q]}\leq\rho_{k_f}^{[p,q]}+\rho_{k_g}^{[p,q]}$.
Again from (1.8)

$$\frac{log^{[p-1]}M_{f^k,D}(\sigma)}{[log^{[q-1]}\sigma]^{\rho_{k_f}^{[p,q]}(D)}}<T_{k_f}^{[p,q]}(D)+\epsilon$$

and

$$\frac{log^{[p-1]}M_{g^k,D}(\sigma)}{[log^{[q-1]}\sigma]^{\rho_{k_g}^{[p,q]}(D)}}<T_{k_g}^{[p,q]}(D)+\epsilon.$$

Now taking product we get,

$$\{\frac{log^{[p-1]}M_{f^k,D}(\sigma)}{[log^{[q-1]}\sigma]^{\rho_{k_f}^{[p,q]}(D)}}\}\{\frac{log^{[p-1]}M_{g^k,D}(\sigma)}{[log^{[q-1]}\sigma]^{\rho_{k_g}^{[p,q]}(D)}}\}<(T_{k_f}^{[p,q]}(D)+\epsilon)(T_{k_g}^{[p,q]}(D)+\epsilon).$$

If

$$[log^{[p-1]}M_{f^k,D}(\sigma)\ log^{[p-1]}M_{g^k,D}(\sigma)]\sim log^{[p-1]}M_{f^k*g^k,D}(\sigma)$$

then we get from above

$$\frac{log^{[p-1]}M_{f^k*g^k,D}(\sigma)}{[log^{[q-1]}\sigma]^{\rho_{k_f}^{[p,q]}(D)+\rho_{k_g}^{[p,q]}(D)}}<(T_{k_f}^{[p,q]}(D)+\epsilon)(T_{k_g}^{[p,q]}(D)+\epsilon).$$

Since $\rho_k^{[p,q]}=\rho_{k_f}^{[p,q]}+\rho_{k_g}^{[p,q]}$ then

$$\limsup_{\sigma\to\infty}\frac{log^{[p-1]}M_{f^k*g^k,D}(\sigma)}{[log^{[q-1]}\sigma]^{\rho_k^{[p,q]}(D)}}\leq T_{k_f}^{[p,q]}(D)T_{k_g}^{[p,q]}(D)$$

i.e.,

$$T_k^{[p,q]}\leq T_{k_f}^{[p,q]}T_{k_g}^{[p,q]}.$$

This completes the proof.

Theorem 2.4. *If*

$$\rho^{[p,q]}(D)=\limsup_{\sigma\to\infty}\frac{log^{[p]}M_{f,D}(\sigma)}{log^{[q]}\sigma}$$

then

$$\limsup_{\sigma\to\infty}\frac{log^{[p]}\mu_{f,D}(l+\sigma)}{log^{[q]}\sigma}\leq\rho^{[p,q]}(D)$$

and $\limsup_{\sigma\to\infty} \frac{log^{[p]}\mu_{f,D}(l+\sigma+\epsilon)}{log^{[q]}\sigma} \geq \frac{\rho^{[p,q]}(D)}{K}$, *where* K *is a positive constant.*

Proof. We have from Theorem 1.1

$$\mu_f(l+\sigma) \leq M_{f,D}(\sigma)$$

$$i.e.,\ \limsup_{\sigma\to\infty} \frac{log^{[p]}\mu_{f,D}(l+\sigma)}{log^{[q]}\sigma} \leq \limsup_{\sigma\to\infty} \frac{log^{[p]}M_{f,D}(\sigma)}{log^{[q]}\sigma}$$

$$So,\ \limsup_{\sigma\to\infty} \frac{log^{[p]}\mu_{f,D}(l+\sigma)}{log^{[q]}\sigma} \leq \rho^{[p,q]}(D).$$

Also

$$K\mu_f(l+\sigma+\epsilon) \geq M_{f,D}(\sigma)$$

$$i.e.,\ \limsup_{\sigma\to\infty} \frac{log^{[p]}\mu_{f,D}(l+\sigma+\epsilon)}{log^{[q]}\sigma} \geq \frac{1}{K} \limsup_{\sigma\to\infty} \frac{log^{[p]}M_{f,D}(\sigma)}{log^{[q]}\sigma}$$

$$So,\ \limsup_{\sigma\to\infty} \frac{log^{[p]}\mu_{f,D}(l+\sigma+\epsilon)}{log^{[q]}\sigma} \geq \frac{\rho^{[p,q]}(D)}{K}.$$

Hence the result.

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